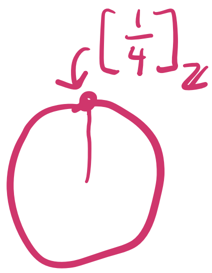
Circle Rotations



$$= \mathbb{R}/\mathbb{Z}$$

$$= \{ [x]_{\mathbb{Z}} : x \in \mathbb{R} \}$$

$$R_\alpha([x]_{\mathbb{Z}}) = [x+\alpha]_{\mathbb{Z}}$$

when is the orbit of a point periodic?

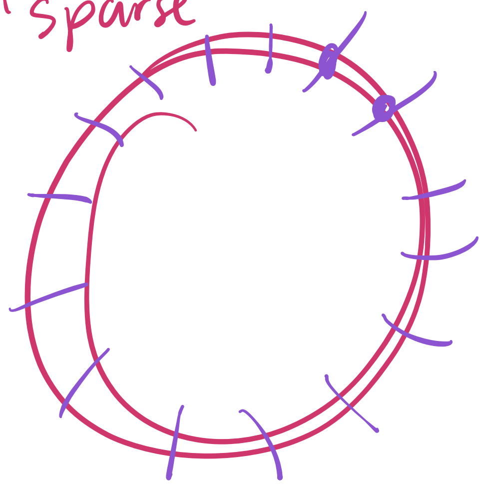
Answer $\alpha := \frac{p}{q}$ $p \in \mathbb{Z}$, $q \in \mathbb{N}$

(ie α is rational)

(ie $\alpha \in \mathbb{Q}$ = the set of rational numbers)

$\boxed{\text{exercise | Find a formula for } \mathbb{Q} \text{ using set builder notation}}$

"sparse"

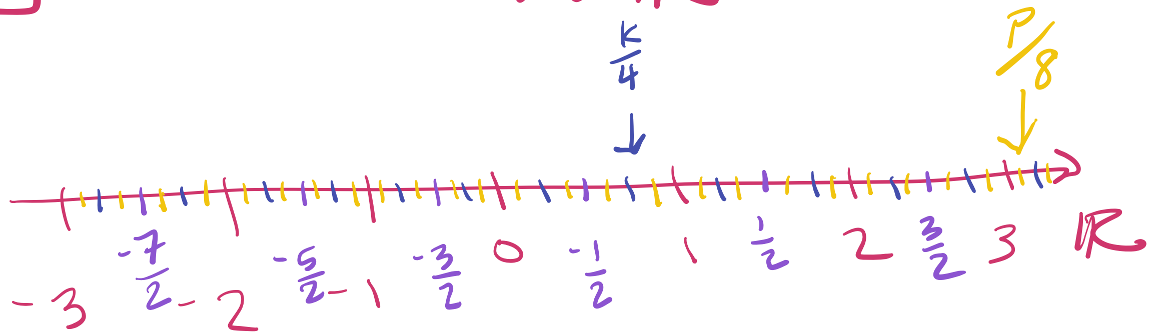


rational rotation gives finite # of points on $\mathbb{R}/\mathbb{Z}$

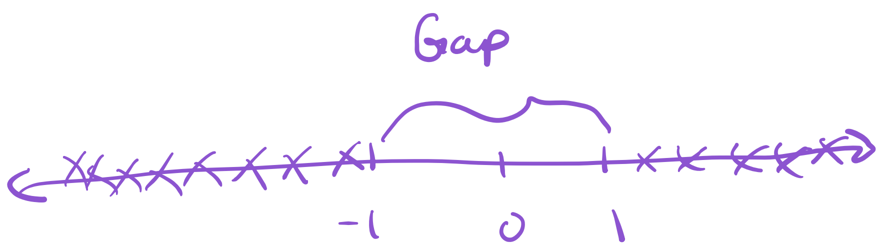
opposite of ~~finite/sparse~~?

On $\mathbb{R}$ what does it mean for a set to be dense?

EX | claim: $\mathbb{Q}$ is dense in $\mathbb{R}$



Ex) $\{x \in \mathbb{Q} : |x| \geq 1\}$ is not dense in $\mathbb{R}$



Gaps between points mean

<u>not</u> dense

<u>Def)</u> A set $A \subset \mathbb{R}$ is <u>not dense</u> if there exists an interval (a,b) such that

$(a,b) \cap A = \emptyset$ {their intersection has no points}

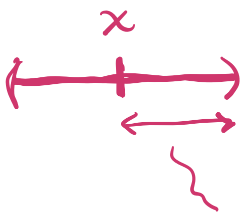
<u>Def)</u> $A \subset \mathbb{R}$ is <u>dense</u> if

for any interval (a,b), $(a,b) \cap A \neq \emptyset$

essential definition!

Thm $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof Pick an interval $(a,b) \subset \mathbb{R}$



let $x = \dfrac{a+b}{2}$

$$\varepsilon = x - a = \frac{b-a}{2} > 0$$

Write x in its base 10 decimal expansion

$$x = x_n x_{n-1} \cdots x_0 . x_{-1} x_{-2} x_{-3} \cdots$$

(eg $x = 937.832156\ldots$)

How to produce a rational # close to x?
cut off decimal expansion

define $x_N = x_n x_{n-1} \cdots x_0 x_{-1} \cdots x_{-N}$.

$$|x - x_N| = |0.\underbrace{00000\ldots0}_{N\ 0\text{'s}}x_{-N-1}x_{-N-2}\cdots|$$

$$< 0.\underbrace{0000\ldots0}_{N-1\ 0\text{'s}}1 = 10^{-(N-1)}$$

No matter how small ε is we can choose

N s.t. $10^{-(N-1)} < \varepsilon$ (since $10^{-(N-1)} \to 0$)

$$|x - x_N| < \varepsilon$$

$$\boxed{\text{exercise} \quad x_N \in (a,b)}$$

$\Rightarrow$ since $x_N \in \mathbb{Q} \cap (a,b)$,

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Tomorrow: ~~define~~ Subgroups

Thm If $H \subset \mathbb{R}$ is a subgroup either

(i) $H = c\mathbb{Z}$ for some c or (ii) H is dense